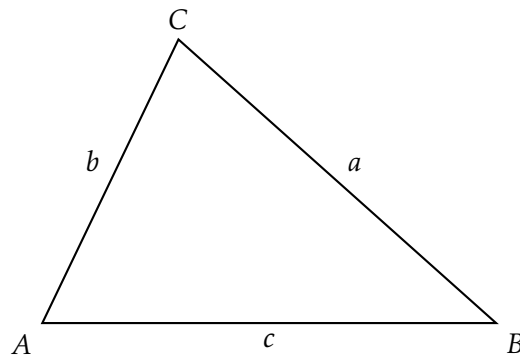


## The Sine Rule

### Things you should already know

- SOHCAHTOA for right-angled triangles
- Solving trig equations over  $0^\circ \leq \theta \leq 360^\circ$
- Angles in a triangle sum to  $180^\circ$

### Labelling Convention



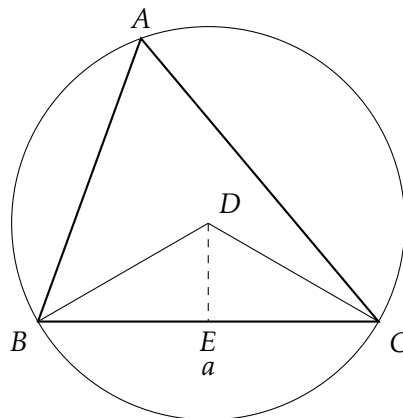
Capital letters for angles, lowercase for the side **opposite** that angle. So side  $a$  is opposite angle  $A$ .

**Fact — The Sine Rule:**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use this when you have a **matching pair** — a side and the angle opposite it.

### Proof



$D$  is the circumcentre (centre of the circle through all three vertices). Let  $R$  be the circumradius.

By the circle theorem (angle at centre = twice angle at circumference):  $\angle BDC = 2A$ .

Triangle  $BDC$  is isosceles ( $BD = CD = R$ ), so  $\angle BDE = A$ .

In  $\triangle BDE$ :  $\frac{\frac{1}{2}a}{R} = \sin A$ , so  $\frac{a}{\sin A} = 2R$ .

Similarly for sides  $b$  and  $c$ , giving  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ .

**Example**

In  $\triangle ABC$ ,  $a = 7$  cm,  $A = 58^\circ$ ,  $B = 23^\circ$ . Find the length  $b$ .

$$\frac{7}{\sin 58^\circ} = \frac{b}{\sin 23^\circ}$$

$$b = \frac{7 \times \sin 23^\circ}{\sin 58^\circ} = 3.23 \text{ cm (3 s.f.)}$$

**Example**

In  $\triangle ABC$ ,  $a = 7$  cm,  $A = 128^\circ$ ,  $b = 6$  cm. Find angle  $B$ .

Using the sine rule “upside down” to find an angle:

$$\frac{\sin B}{6} = \frac{\sin 128^\circ}{7}$$

$$\sin B = \frac{6 \times \sin 128^\circ}{7} = 0.6752\dots$$

$$B = \sin^{-1}(0.6752) = 42.5^\circ \text{ (3 s.f.)}$$

Note:  $180^\circ - 42.5^\circ = 137.5^\circ$  is impossible since  $A = 128^\circ$  is already obtuse.

**Textbook Exercises: SPSC: 4.10** — Exercise 3A, questions 1–10.

**Dr Frost:** Sine and Cosine Rules slides — sine rule section.

**Corbett Maths:** Sine Rule practice questions (Video 314).

**Maths Genie:** Sine Rule worksheet.

**Maths4Everyone:** Sine Rule booklet.

## The Cosine Rule

### Example

In  $\triangle ABC$ ,  $b = 8$  cm,  $c = 5$  cm,  $A = 60^\circ$ . Can you use the sine rule? Why or why not?

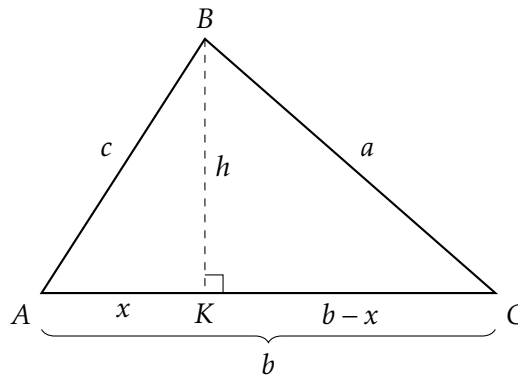
No — we don't have a "matching pair". We have two sides and the **included** angle. We need the cosine rule.

### Fact — The Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Use this when you have **two sides and the included angle** (SAS), or **all three sides** (SSS).

### Proof



Drop a perpendicular from B to AC, meeting at K. Let  $BK = h$  and  $AK = x$ .

In  $\triangle ABK$ :  $h^2 = c^2 - x^2$

In  $\triangle BCK$ :  $h^2 = a^2 - (b-x)^2$

Setting equal:  $c^2 - x^2 = a^2 - (b-x)^2$

Expanding:  $c^2 - x^2 = a^2 - b^2 + 2bx - x^2$

So:  $a^2 = b^2 + c^2 - 2bx$

But  $x = c \cos A$ , giving  $a^2 = b^2 + c^2 - 2bc \cos A$ . □

**Finding a side (SAS)****Example**

In  $\triangle ABC$ ,  $b = 9$  cm,  $a = 8$  cm,  $C = 127^\circ$ . Find  $c$ .

$$\begin{aligned}c^2 &= a^2 + b^2 - 2ab \cos C \\&= 8^2 + 9^2 - 2(8)(9) \cos 127^\circ \\&= 64 + 81 - 144 \times (-0.6018\dots) \\&= 145 + 86.66\dots = 231.66\dots\end{aligned}$$

$$c = \sqrt{231.66} = 15.2 \text{ cm (3 s.f.)}$$

*NB: Don't forget to take the square root!*

**Example**

In  $\triangle PQR$ ,  $p = 11$  cm,  $r = 7$  cm,  $Q = 48^\circ$ . Find  $q$ .

$$\begin{aligned}q^2 &= p^2 + r^2 - 2pr \cos Q \\&= 11^2 + 7^2 - 2(11)(7) \cos 48^\circ \\&= 121 + 49 - 154 \times 0.6691\dots \\&= 170 - 103.04\dots = 66.96\dots\end{aligned}$$

$$q = \sqrt{66.96} = 8.18 \text{ cm (3 s.f.)}$$

## Finding an angle (SSS)

### Example

In  $\triangle ABC$ ,  $a = 8$ ,  $b = 9$ ,  $c = 10$ . Find the largest angle.

The largest angle is opposite the longest side, so we find  $C$ :

$$c^2 = a^2 + b^2 - 2ab \cos C, \text{ so } 10^2 = 8^2 + 9^2 - 2(8)(9) \cos C$$

$$100 = 145 - 144 \cos C$$

$$\cos C = \frac{145 - 100}{144} = \frac{45}{144} = 0.3125$$

$$C = \cos^{-1}(0.3125) = 71.8^\circ \text{ (3 s.f.)}$$

### Example

In  $\triangle ABC$ ,  $a = 5$ ,  $b = 6$ ,  $c = 9$ . Find angle  $C$ .

$$c^2 = a^2 + b^2 - 2ab \cos C, \text{ so } 9^2 = 5^2 + 6^2 - 2(5)(6) \cos C$$

$$81 = 61 - 60 \cos C$$

$$\cos C = \frac{61 - 81}{60} = \frac{-20}{60} = -\frac{1}{3}$$

$$C = \cos^{-1}\left(-\frac{1}{3}\right) = 109.5^\circ \text{ (1 d.p.)}$$

Note:  $\cos C < 0$ , so  $C$  is obtuse — this makes sense since  $c$  is much larger than  $a$  and  $b$ .

**Textbook Exercises: SPSC: 4.10** — Exercise 3A, questions 11–15.

**Dr Frost:** Sine and Cosine Rules slides — cosine rule section.

**Corbett Maths:** Cosine Rule practice questions (Video 81).

**Maths Genie:** Cosine Rule worksheet.

**Maths4Everyone:** Cosine Rule booklet.

**Area of a Triangle:**  $\frac{1}{2}ab \sin C$ **Example**

You know two sides and the included angle of a triangle. Can you find its area without finding the height first?

**Fact — Area of a triangle:**

$$\text{Area} = \frac{1}{2}ab \sin C$$

where  $a$  and  $b$  are two sides and  $C$  is the **included angle** between them.

**Proof**

Using the same diagram as the cosine rule proof: the height  $h = a \sin C$ .

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times b \times a \sin C = \frac{1}{2}ab \sin C. \quad \square$$

**Example**

Find the area of  $\triangle ABC$  where  $a = 7$  cm,  $b = 8.5$  cm,  $C = 72^\circ$ .

$$\text{Area} = \frac{1}{2}(7)(8.5) \sin 72^\circ = \frac{1}{2} \times 59.5 \times 0.9511 = 28.3 \text{ cm}^2 \text{ (3 s.f.)}$$

**Example**

Find the area of  $\triangle ABC$  where  $a = 12.5$  cm,  $A = 48^\circ$ ,  $B = 65^\circ$ .

First find  $C = 180^\circ - 48^\circ - 65^\circ = 67^\circ$ .

Use the sine rule to find  $b$ :

$$\frac{b}{\sin 65^\circ} = \frac{12.5}{\sin 48^\circ} \implies b = \frac{12.5 \sin 65^\circ}{\sin 48^\circ} = 15.25 \text{ cm}$$

$$\text{Area} = \frac{1}{2}(12.5)(15.25)\sin 67^\circ = 87.8 \text{ cm}^2 \text{ (3 s.f.)}$$

## Choosing the Right Rule

### Fact — Decision flowchart:

1. Do you have a **matching pair** (side + opposite angle)?
  - Yes → **Sine Rule**
2. Do you have **two sides + included angle** (SAS)?
  - Yes → **Cosine Rule** (to find the third side)
3. Do you have **three sides** (SSS)?
  - Yes → **Cosine Rule** (to find an angle)
4. Always draw a diagram!
5. You may need to use  $A + B + C = 180^\circ$ .

**Textbook Exercises: SPSC:** 4.10 — Exercise 3A, questions 16–29.

**Dr Frost:** Sine and Cosine Rules slides — area of a triangle and mixed problems.

**Corbett Maths:** Area of a Triangle using Sine practice questions (Video 23b).

**Maths Genie:** Area of a Triangle worksheet (non-right-angled).

**Maths4Everyone:** Area of a Triangle (Trig) booklet.

## The Ambiguous Case of the Sine Rule

### Example

In  $\triangle ABC$ ,  $a = 10$  cm,  $b = 7$  cm,  $A = 40^\circ$ . Find angle  $B$ .

$$\frac{\sin B}{7} = \frac{\sin 40^\circ}{10} \implies \sin B = \frac{7 \sin 40^\circ}{10} = 0.4499\dots$$

$$B = \sin^{-1}(0.4499) = 26.7^\circ \text{ (3 s.f.)}$$

But  $\sin B = 0.4499$  also gives  $B = 180^\circ - 26.7^\circ = 153.3^\circ$ .

Check:  $A + B = 40^\circ + 153.3^\circ = 193.3^\circ > 180^\circ$  — impossible!

So only  $B = 26.7^\circ$  works here.

### Example

In  $\triangle ABC$ ,  $a = 5$  cm,  $b = 8$  cm,  $A = 30^\circ$ . Find angle  $B$ .

$$\frac{\sin B}{8} = \frac{\sin 30^\circ}{5} \implies \sin B = \frac{8 \times 0.5}{5} = 0.8$$

$$B = \sin^{-1}(0.8) = 53.1^\circ \quad \text{or} \quad B = 180^\circ - 53.1^\circ = 126.9^\circ$$

Check:  $A + B = 30^\circ + 53.1^\circ = 83.1^\circ < 180^\circ$  ✓

Check:  $A + B = 30^\circ + 126.9^\circ = 156.9^\circ < 180^\circ$  ✓

**Both values are valid** — there are **two possible triangles**.

This is the **ambiguous case**: it arises when you are given two sides and an angle **not** between them (SSA).

**Fact** — The ambiguous case occurs when using the sine rule to find an angle. Since  $\sin \theta = \sin(180^\circ - \theta)$ ,

there may be **two valid solutions**. Always check whether the second solution gives a valid triangle (angles summing to less than  $180^\circ$ ).